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STOCHASTIC SYSTEMS

Multivariate Continuous Distributions and Copulas Generating Nontransitive Tuples of Dependent Random Variables

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Abstract—This paper continues the author's research series on the nontransitivity phenomenon of the stochastic precedence relation in probability theory. Based on the Condorcet paradox, examples of trivariate continuous distributions and copulas generating nontransitive tuples of dependent random variables are constructed. Limit theorems for multivariate mixtures are proven.

Keywords: nontransitivity, stochastic precedence, copulas, multivariate mixtures, limit theorems

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1. INTRODUCTION

In theory and practice, various superiority relations between objects often have the property of transitivity: if A is superior to B and B is superior to C, then A is superior to C. However, it happens the other way around. For example, in the classical rock–paper–scissors game (also known as Rochambeau, Roshambo, Ro-sham-bo, Bato Bato Pik, and Jak-en-poy in some countries), rock beats scissors, scissors beats paper but, at the same time, paper beats rock.

Several articles by A.N. Poddiakov, particularly [1, 2], were devoted to various aspects and numerous examples of the nontransitivity of superiority relations in nature, technology, and society.

Among recent publications, note the interesting studies of nontransitivity in biology, from the life of ants [3] and soil bacteria [4].

Below, we will address the nontransitivity of the stochastic precedence relation between random variables, i.e., when one variable is more likely to be smaller than the other. This relation was applied in statistical analysis problems [5, 6]. Methods for comparing probability distributions based on statistical data (histograms) were the subject of the recent work [7], where the author considered various stochastic orders.

H. Steinhaus and S. Trybula first mentioned the problem of nontransitivity from the probabilistic point of view [8, 9], with the strength of materials as an application. Let the strength of iron bars from three different factories be compared pairwise in laboratory conditions. Theoretically, there may be a situation that bars from the first factory are worse than those from the second (they have lower strength more often) and, simultaneously, bars from the second factory are worse than those from the third and bars from the third factory are worse than those from the first.

Recently, such a phenomenon has been discovered in medical statistics when comparing life expectancy in some groups of people [10].

The issue of nontransitivity has become widespread on an example of *nontransitive dice*. In such dice sets, the numbers on faces are appropriately drawn to create the nontransitive relations of the corresponding random variables. Nontransitive dice were popularized by M. Gardner [11, 12]; see the extensive literature in this field.

In this paper, we continue the studies of nontransitivity initiated in [13–17]. (The interested reader can find more details about the range of problems, historical background, related literature, and applications therein.) In contrast to the earlier publications focused on the case of independent random variables, below we consider dependent random variables.

In this sense, a classical example is the famous Condorcet paradox of voting. Let us present its simplest formulation. There are three candidates for some post and three voters who have individual preferences for candidates (in points) according to the following row vectors:

$$\begin{aligned}&(1,2,3),\\&(3,1,2),\\&(2,3,1).\end{aligned}\tag{1}$$

Then, when choosing between the first and second candidates, the second will win. Indeed, the first and third voters have higher points for the second candidate than for the first ($1 < 2$ and $2 < 3$, respectively), and the second voter has lower points ($3 > 1$); as a result, the second candidate will win by two votes out of three. Similarly, when choosing between the second and third candidates, the third will win; between the first and third, the first.

Such situations often arise when experts assess, e.g., the quality of goods. In this case, difficulties are because many characteristics cannot be directly measured in quantitative terms (by technical means) and are assessed intuitively in conventional units (points). In the case of food products, an important role is played by organoleptic indicators: taste, color, smell, etc. It is necessary to exclude nontransitive subsets in expert measurements, and various methods are used for this purpose [18–20].

As is believed, increasing the number of experts (interviewees) solves the problem; however, sometimes this approach does not work. For example, a mass survey on the choice of a territory development project (green zone, family recreation park, business center) revealed nontransitivity in answers [21]. It is assumed that people assess projects considering certain factors (environmental, social, financial), but the weights (importance) of these factors vary for different people. If the joint distribution of weights were similar to those studied below, this would explain the results. In addition, a more detailed analysis of the data using computer statistical methods (cluster analysis, factor analysis, etc.) would show the division of people into groups based on their views, even if these groups are not formally positioned or realized by the participants.

Note also the works [22, 23].

The properties of decision-making methods in multicriteria individual choice problems were extensively studied in [24]. Under nontransitivity conditions, a “best” alternative can always be found when comparing some current alternative with others, which leads to an infinite Markov random walk in formal terms. In the modern development of artificial intelligence, automatic decision systems, and robots, it is necessary to consider the nontransitivity phenomenon. A human can realize such a situation and make a willed decision whereas a machine can get stuck.

2. DEFINITIONS AND KNOWN RESULTS

We begin with strict definitions.

Definition 1. A relation $\prec$ is said to be nontransitive if, for any objects A, B, and C, the relations $A \prec B$ and $B \prec C$ DO NOT imply the relation $A \prec C$ but, on the contrary, it is possible that $C \prec A$.

Definition 2. A set of objects A, B, and C is said to be nontransitive if the nontransitivity of the superiority relation is realized on it, i.e., $A \prec B \prec C \prec A$ (or in the reverse order).

Definition 3. X stochastically precedes Y $(X \prec Y)$ if

$$\mathbf{E}\,\mathrm{sgn}(Y - X) > 0$$

or

$$\mathbf{P}(X < Y) > \mathbf{P}(X > Y).$$

If $\mathbf{P}(X = Y) = 0$ (e.g., the random variables are independent and continuous or have nonoverlapping value sets), then $X \prec Y$ is equivalent to

$$\mathbf{P}(X < Y) > \frac{1}{2}.$$

Note that if random variables are independent and continuous, and obey an identical distribution, then the equality $\mathbf{P}(X < Y) = 1/2$ always holds. For continuous random variables, this equality can be violated due to different distributions or dependence; both cases require separate consideration.

Well, let random variables X_1, X_2, and X_3 be such that

$$\mathbf{P}(X_1 < X_2) > \frac{1}{2}, \quad \mathbf{P}(X_2 < X_3) > \frac{1}{2}, \quad \mathbf{P}(X_3 < X_1) > \frac{1}{2},$$

then $X_1 \prec X_2$, $X_2 \prec X_3$, but $X_3 \prec X_1$. Thus, a situation of nontransitivity arises when

$$P_{X_1X_2X_3} = \min\{\mathbf{P}(X_1 < X_2), \mathbf{P}(X_2 < X_3), \mathbf{P}(X_3 < X_1)\} > \frac{1}{2},$$

and not only the fact but also strength of nontransitivity is of interest. The latter can be measured by the value of $P_{X_1X_2X_3}$.

According to [9], for independent random variables,

$$\max_{X_1,X_2,X_3} P_{X_1X_2X_3} = \frac{\sqrt{5}-1}{2} \approx 0.618. \tag{2}$$

A combination of $n \geqslant 3$ independent random variables was studied in [26, 27]. As was subsequently shown in [28] (and later reproved geometrically in [29]), the maximum of the probabilities

$$P_{X_1 \ldots X_n} = \min\{\mathbf{P}(X_1 < X_2), \ldots, \mathbf{P}(X_{n-1} < X_n), \mathbf{P}(X_n < X_1)\}$$

is

$$\max_{X_1,\ldots,X_n} P_{X_1 \ldots X_n} = 1 - \left(4\cos^2 \frac{\pi}{n+2}\right)^{-1}, \quad n \geqslant 3.$$

This result coincides with (2) for $n = 3$ since

$$\cos \frac{\pi}{5} = \frac{1+\sqrt{5}}{4}.$$

In the case of dependent variables [27],

$$\max_{X_1,\ldots,X_n} P_{X_1 \ldots X_n} = \frac{n-1}{n}, \quad n \geqslant 3;$$

particularly for $n = 3$, we obtain $P_{X_1X_2X_3} = 2/3$.

For three dependent random variables, a game interpretation can be as follows: a random vector (X_1, X_2, X_3) is played after two players have sequentially chosen a component of this vector. The winner is the player whose component takes a higher value. Nontransitivity means that whatever component the first player chooses, the second can choose his/her component so that the probability of win be $P_{X_1X_2X_3} > 1/2$, i.e., get an advantage.

Consider a probabilistic-statistical modification of the Condorcet paradox corresponding to an arbitrarily large number of voters, each having one of the three opinions about the candidates described by (1) equiprobably.

Let the set of dependent random variables (X, Y, Z) take values from (1) with probability 1/3 each. Then

$$\mathbf{P}(X < Y) = \mathbf{P}(Y < Z) = \mathbf{P}(Z < X) = \frac{2}{3}.$$

In this case, each of the variables is uniformly distributed on the set $\{1, 2, 3\}$.

The question arises: how can we pass from a discrete distribution to its continuous counterpart? Consider different examples.

Recall the concept of a copula [30].

Definition 4. An (m-dimensional) copula C is a multivariate distribution function on $[0, 1]^m$ with the uniform partial (marginal) distributions.

A copula of a distribution F in R^m is a copula C that satisfies the expression

$$F(x_1, \ldots, x_m) = C(F_1(x_1), \ldots, F_m(x_m)),$$

where $F_1, \ldots, F_m$ are marginal distribution functions.

Such a representation exists by Sklar's theorem and is unique in the case of continuous marginal distributions. From this point onwards, we assume continuity.

If $X_1, \ldots, X_m$ are given and $U_i = F_i(X_i)$, $1 \leqslant i \leqslant n$, then all U_i are uniformly distributed on $[0, 1]$, with the joint distribution C, i.e.,

$$\mathbf{P}(U_1 \leqslant u_1, \ldots, U_m \leqslant u_m) = C(u_1, \ldots, u_m).$$

In the continuous case with all X_i identically distributed, we have

$$\mathbf{P}(X_i < X_j) = \mathbf{P}(U_i < U_j), \quad i \neq j.$$

3. THREE-DIMENSIONAL CONTINUOUS COPULAS

Consider first some natural generalizations of the Condorcet example with uniform marginal distributions on $[0, 1]$. (The resulting trivariate distributions can be used as copulas.)

Example 1. Let

$$X = T, \quad Y = \left\{T + \frac{1}{3}\right\}, \quad Z = \left\{T + \frac{2}{3}\right\},$$

where $\{.\}$ denotes the fractional part of a number and T is uniformly distributed on $[0, 1]$. Then X, Y, and Z obey the uniform distribution on $[0, 1]$ and

$$\mathbf{P}(X < Y) = \mathbf{P}(Y < Z) = \mathbf{P}(Z < X) = \frac{2}{3}. \tag{3}$$

The graphs of X, Y, and Z depending on T are shown in Fig. 1. The probabilities correspond to the fractions of the closed interval $[0, 1]$ on which the corresponding inequalities are true.

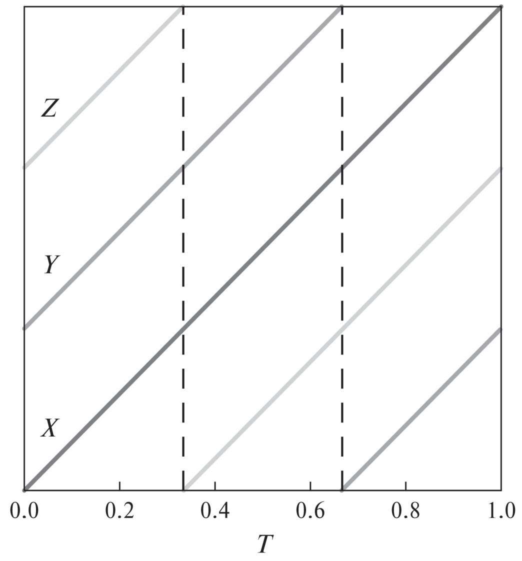

Fig. 1. Example 1: X, Y, and Z depending on T.

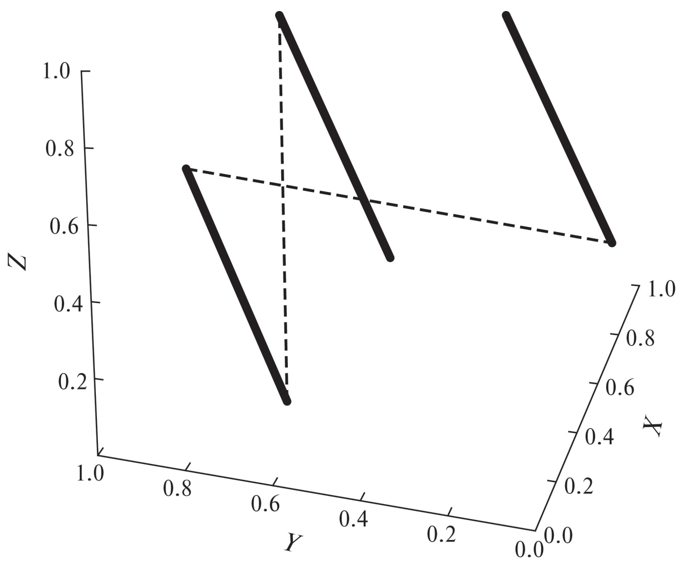

Fig. 2. Example 1: the distribution in the unit cube.

In this case, the distribution of the vector turns out to be continuous, albeit not absolutely continuous (having a density) but singular (centered on a manifold of lower dimension, i.e., three segments) in the unit cube; see Fig. 2.

Now we proceed to absolutely continuous distributions. Let $p(x, y, z)$, where $x, y, z \in [0, 1]$, denote the joint distribution density of random variables X, Y, and Z.

Example 2. Let $p(x, y, z) = 2$ if $x < y < z$ or $z < x < y$ or $y < z < x$, and $p(x, y, z) = 0$ otherwise. Then X, Y, and Z are also uniformly distributed on $[0, 1]$ and formula (3) is valid. The above density can be represented as

$$p(x, y, z) = 1 - \operatorname{sgn}\{(y - x)(z - y)(x - z)\}. \tag{4}$$

This distribution is centered in three pyramids inside the unit cube (Fig. 3).

In this case, the trivariate distribution has a density, but this density is discontinuous. However, the form of (4) suggests to pass to polynomial density.

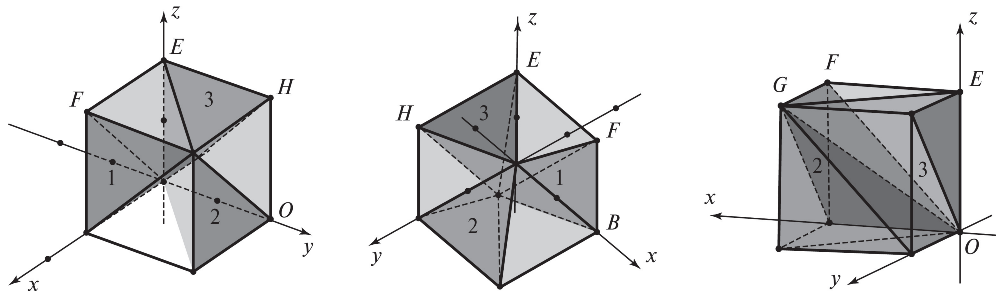

Fig. 3. Example 2: the distribution in the unit cube.

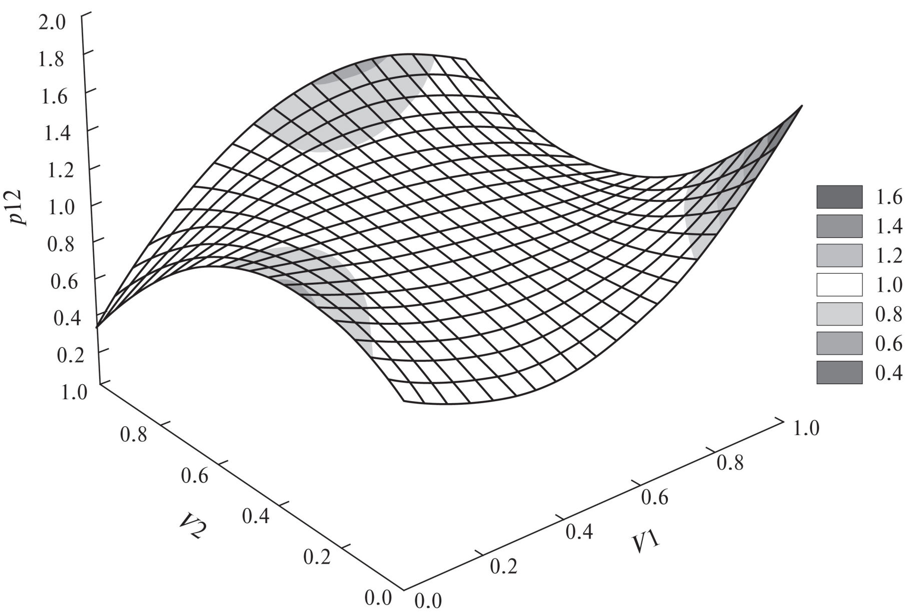

Fig. 4. Example 3: the bivariate distribution density.

Example 3. Let

$$p(x, y, z) = 1 - K(y - x)(z - y)(x - z). \tag{5}$$

As can be verified, in this case, the marginal distributions remain uniform on $[0, 1]$. The question is to choose a constant K so that the density be nonnegative.

We have

$$\max_{[0,1]^3}\{(y - x)(z - y)(x - z)\} = \frac{1}{4},$$

and this maximum is achieved at the points $(1/2, 1, 0)$, $(0, 1/2, 1)$, and $(1, 0, 1/2)$.

Hence, the maximal admissible value is $K = 4$; with

$$p(x, y, z) = 1 - 4(y - x)(z - y)(x - z),$$

integration yields the joint distribution function

$$F(x, y, z) = xyz\left(1 + \frac{2}{3}(xy(y - x) + yz(z - y) + xz(x - z))\right).$$

The joint density of X and Y takes the form

$$p_{12}(x, y) = 1 + \frac{2}{3}(y - x)(2 - 3x - 3y + 6xy)$$

(see Fig. 4); therefore,

$$\mathbf{P}(X < Y) = \mathbf{P}(Y < Z) = \mathbf{P}(Z < X) = \frac{47}{90} = 0.522\ldots.$$

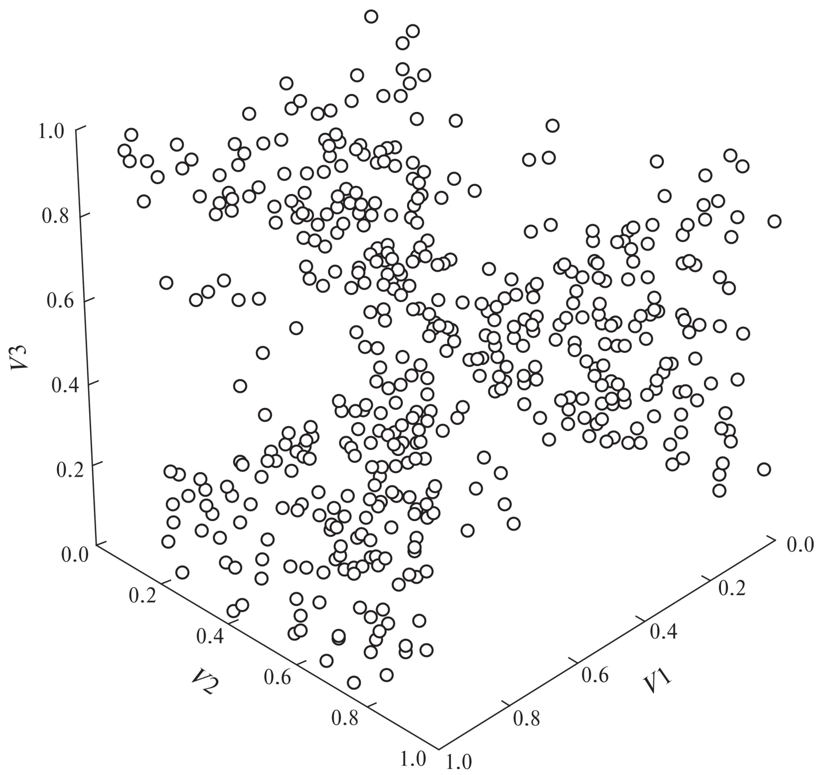

Fig. 5. Example 4: the simulation result.

Example 4. Consider an intermediate density between (4) and (5):

$$p(x, y, z) = 1 - \sqrt[k]{4(y - x)(z - y)(x - z)}, \tag{6}$$

where $k \geqslant 3$ is an odd natural number. Obviously, as $k \to \infty$, we have convergence to the situation described in Example 2. The probability P_{XYZ} was estimated by Monte Carlo simulations with 10^6 points; see the table below.

k	P_{XYZ}
3	0.569
5	0.595
7	0.610
9	0.621
11	0.628

Figure 5 presents the simulation result for $k = 11$ and 10^3 points. The convergence to the three pyramids from Example 2 can be observed.

4. MULTIVARIATE MIXTURES

Now consider another approach, with noise introduced into discrete data, as in [31] as applied to nontransitive Efron dice (independent random variables).

Example 5. Let a random vector (X_1, X_2, X_3) be formed as follows: independent Gaussian variables with zero mean and a variance $\sigma^2 > 0$ are added coordinate-wise to the values of (1), taken with equal probabilities 1/3.

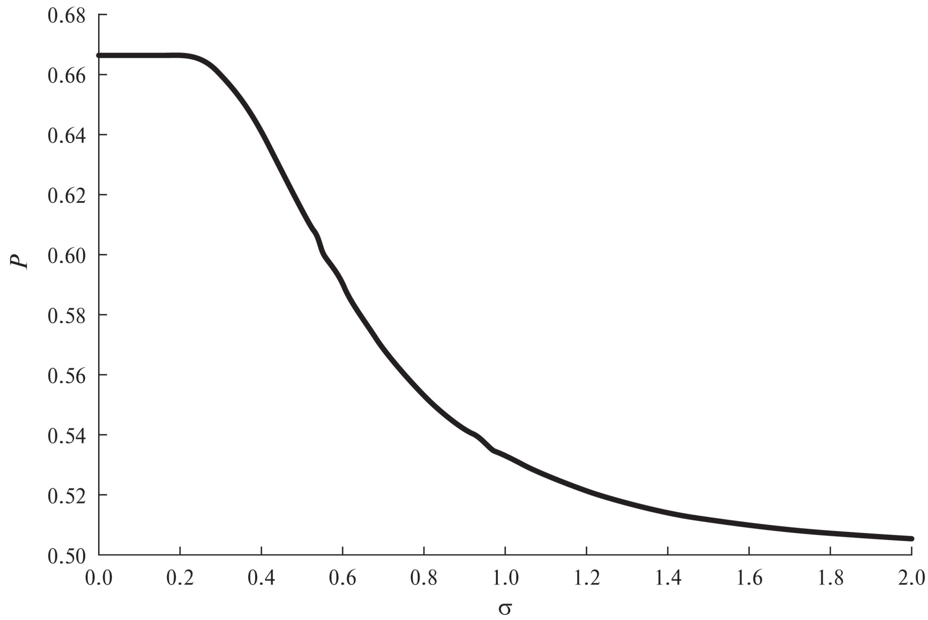

Fig. 6. Example 5: $P_{X_1X_2X_3}$ depending on σ.

Proposition 1. *In Example 5,*

$$P_{X_1X_2X_3} = \mathbf{P}(X_1 < X_2) = \mathbf{P}(X_2 < X_3) = \mathbf{P}(X_3 < X_1) = \frac{1}{3}\left(2\Phi\left(\frac{1}{\sigma\sqrt{2}}\right) + \Phi\left(-\frac{\sqrt{2}}{\sigma}\right)\right) > \frac{1}{2},$$

where $P_{X_1X_2X_3} \to 2/3$ *as* $\sigma \to 0$ *and* $P_{X_1X_2X_3} \to 1/2$ *as* $\sigma \to \infty$.

The proof is given in the Appendix.

Here, nontransitivity is observed for *any* $\sigma^2 > 0$.

The graph of the probability $P_{X_1X_2X_3}$ depending on σ is shown in Fig. 6.

The following question arises: as $\sigma \to 0$, the distribution converges to a discrete (three-point) distribution, and what happens to the copula?

Proposition 2. *In Example 5, as* $\sigma \to 0$, *the distribution of the copula converges to the uniform distribution on the union of cubes*

$$[0, 1/3] \times [1/3, 2/3] \times [2/3, 1],$$
$$[2/3, 1] \times [0, 1/3] \times [1/3, 2/3],$$
$$[1/3, 2/3] \times [2/3, 1] \times [0, 1/3],$$

with the density $p(x_1, x_2, x_3) = 9$ *on this set and* 0 *otherwise.*

The proof is given in the Appendix.

With the standard uniform distribution function denoted by

$$R(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leqslant x < 1 \\ 1, & x \geqslant 1, \end{cases} \tag{7}$$

the limit copula from Proposition 2 can be represented as

$$C_0(u_1, u_2, u_3) = \frac{1}{3}\Big(R(3u_1)R(3u_2 - 1)R(3u_3 - 2) + R(3u_1 - 2)R(3u_2)R(3u_3 - 1) + R(3u_1 - 1)R(3u_2 - 2)R(3u_3)\Big).$$

For the limit copula, we have

$$\mathbf{P}(U_1 < U_2) = \mathbf{P}(U_2 < U_3) = \mathbf{P}(U_3 < U_1) = 2/3.$$

Let us proceed to the more general case.

Recall that a (discrete) *multivariate mixture* is a distribution of a random vector that takes, with some probabilities (weights), values of random vectors with given distributions. (Moreover, the vector is chosen independently of the values.)

A mixture of Gaussian vectors is called a *Gaussian* mixture. Thus, Example 5 illustrates a Gaussian mixture.

Let a random vector (X_1, X_2, X_3) be distributed as a mixture, with weights p_i, $i = 1, 2, 3$, of random vectors

$$(a_{i1}, a_{i2}, a_{i3}) + \lambda(\eta_{i1}, \eta_{i2}, \eta_{i3}),$$

where $A = (a_{ij})_{i,j=1}^n$ is some matrix, $(\eta_{i1}, \eta_{i2}, \eta_{i3})$, $i = 1, 2, 3$, are random vectors, and $\lambda > 0$.

In addition, let $0 < p_i < 1/2$, $i = 1, 2, 3$.

Theorem 1. *Assume that $\eta_{i1} - \eta_{i2}$, $\eta_{i2} - \eta_{i3}$, and $\eta_{i3} - \eta_{i1}$ are continuous for all $i = 1, 2, 3$ and a_{ik} in the rows of A have the same order as in (1). Then there exists a number $\lambda_1 > 0$ such that nontransitivity holds for $\lambda < \lambda_1$, i.e., $P_{X_1X_2X_3} > 1/2$, and*

$$\lim_{\lambda \to 0} P_{X_1X_2X_3} = 1 - \max\{p_1, p_2, p_3\}.$$

The proof is given in the Appendix.

Theorem 2. *Assume that $\eta_{ij} = \eta_j$, where η_j are independent and obey an identical continuous distribution, $i, j = 1, 2, 3$, and a_{ik} in the columns of A have the same order as in (1). Then there exists a number $\lambda_2 > 0$ such that the nontransitivity of (U_1, U_2, U_3) holds for $\lambda < \lambda_2$, i.e., $P_{U_1U_2U_3} > 1/2$; moreover, as $\lambda \to 0$, the distribution of the copula converges to a mixture of uniform distributions with weights p_i, $i = 1, 2, 3$, on the parallelepipeds*

$$\begin{gathered}
[0, p_1] \times [p_1, p_1 + p_2] \times [p_1 + p_2, 1], \\
[p_1 + p_2, 1] \times [0, p_1] \times [p_1, p_1 + p_2], \\
[p_1, p_1 + p_2] \times [p_1 + p_2, 1] \times [0, p_1],
\end{gathered}$$

so that

$$\lim_{\lambda \to 0} P_{U_1U_2U_3} = 1 - \max\{p_1, p_2, p_3\}.$$

The proof is given in the Appendix.

In this case, using the notation (7), the limit copula can be written as

$$\begin{aligned}
&C_0(u_1, u_2, u_3) \\
&= p_1 R\left(\frac{u_1}{p_1}\right) R\left(\frac{u_2 - p_1}{p_2}\right) R\left(\frac{u_3 - (p_1 + p_2)}{p_3}\right) \\
&+ p_2 R\left(\frac{u_1 - (p_1 + p_2)}{p_3}\right) R\left(\frac{u_2}{p_1}\right) R\left(\frac{u_3 - p_1}{p_2}\right) \\
&+ p_3 R\left(\frac{u_1 - p_1}{p_2}\right) R\left(\frac{u_2 - (p_1 + p_2)}{p_3}\right) R\left(\frac{u_3}{p_1}\right).
\end{aligned}$$

5. CONCLUSIONS

In this paper, we have established new results regarding the nontransitivity of the stochastic precedence relation in probability theory. Based on the Condorcet paradox, several examples of trivariate continuous distributions and copulas generating nontransitive sets of dependent random variables have been constructed. The cases of singular distributions and distributions with a discontinuous density or polynomial density, as well as the case of multivariate mixtures, have been considered. Limit theorems for multivariate mixtures have been proven. Such multivariate continuous distributions can describe and explain the manifestations of nontransitivity in psychology, economics, biology, and other disciplines. This phenomenon should be taken into account when designing automatic decision systems, artificial intelligence systems, robots, etc.

APPENDIX

Proof of Proposition 1. We use the representation

$$X_i = X_i^0 + \varepsilon_i, \quad i = 1, 2, 3,$$

where the vector (X_1^0, X_2^0, X_3^0) takes the values (1) equiprobably and the random variables $\varepsilon_i \sim N(0, \sigma^2)$ are independent.

Let us calculate the probability $\mathbf{P}(X_1 < X_2)$; the rest can be found by analogy. We have

$$\mathbf{P}(X_1 < X_2) = \mathbf{P}(X_1^0 + \varepsilon_1 < X_2^0 + \varepsilon_2) = \mathbf{P}(\varepsilon_1 - \varepsilon_2 < X_2^0 - X_1^0).$$

With the notation $\varepsilon = \varepsilon_1 - \varepsilon_2$, it follows that $\varepsilon \sim N(0, 2\sigma^2)$ and

$$\begin{aligned}\mathbf{P}(X_1 < X_2) &= \frac{1}{3}\mathbf{P}(\varepsilon < 2 - 1) + \frac{1}{3}\mathbf{P}(\varepsilon < 1 - 3) + \frac{1}{3}\mathbf{P}(\varepsilon < 3 - 2) \\ &= \frac{2}{3}\mathbf{P}(\varepsilon < 1) + \frac{1}{3}\mathbf{P}(\varepsilon < -2) = \frac{1}{3}\left(2\Phi\left(\frac{1}{\sigma\sqrt{2}}\right) + \Phi\left(-\frac{\sqrt{2}}{\sigma}\right)\right).\end{aligned}$$

Proof of Proposition 2. Let ν denote the number of a vector from (1) chosen as the value (X_1^0, X_2^0, X_3^0). Under the condition $\nu = 1$, we have

$$\begin{aligned}U_1 &= \frac{1}{3}\left(\Phi\left(\frac{1 + \varepsilon_1 - 1}{\sigma}\right) + \Phi\left(\frac{1 + \varepsilon_1 - 2}{\sigma}\right) + \Phi\left(\frac{1 + \varepsilon_1 - 3}{\sigma}\right)\right), \\ U_2 &= \frac{1}{3}\left(\Phi\left(\frac{2 + \varepsilon_2 - 1}{\sigma}\right) + \Phi\left(\frac{2 + \varepsilon_2 - 2}{\sigma}\right) + \Phi\left(\frac{2 + \varepsilon_2 - 3}{\sigma}\right)\right), \\ U_3 &= \frac{1}{3}\left(\Phi\left(\frac{3 + \varepsilon_3 - 1}{\sigma}\right) + \Phi\left(\frac{3 + \varepsilon_3 - 2}{\sigma}\right) + \Phi\left(\frac{3 + \varepsilon_3 - 3}{\sigma}\right)\right).\end{aligned}$$

Note that $\varepsilon_i/\sigma \sim N(0, 1)$. We define the random variables $U_i^0 = \Phi(\varepsilon_i/\sigma)$; they are uniformly distributed on $[0, 1]$ and independent. In addition,

$$\Phi\left(\frac{\varepsilon_i + c}{\sigma}\right) \xrightarrow{P} \begin{cases} 0, & c < 0 \\ U_i^0, & c = 0 \\ 1, & c > 0, \end{cases} \qquad \sigma \to 0.$$

Consequently,

$$(U_1, U_2, U_3) \xrightarrow{P} \left(\frac{1}{3}U_1^0, \frac{1}{3} + \frac{1}{3}U_2^0, \frac{2}{3} + \frac{1}{3}U_3^0\right), \quad \sigma \to 0,$$

i.e., we obtain the uniform distribution on the cube $[0,1/3] \times [1/3,2/3] \times [2/3,1]$. The cases $\nu = 2,3$ are considered by analogy.

Proof of Theorem 1. We use the representation

$$(X_1, X_2, X_3) = (a_{\nu 1}, a_{\nu 2}, a_{\nu 3}) + \lambda(\eta_{\nu 1}, \eta_{\nu 2}, \eta_{\nu 3}),$$

where ν takes values 1, 2, and 3 with probabilities p_1, p_2, and p_3, respectively, independently of η_{ij}, $i,j = 1,2,3$.

Let G_i, $i = 1,2,3$, denote the distributions of $\eta_{i1} - \eta_{i2}$.

We have

$$\mathbf{P}(X_1 < X_2) = \sum_{i=1}^{3} p_i G_i \left(\frac{a_{i2} - a_{i1}}{\lambda} \right) \to p_1 + p_3 = 1 - p_2 > \frac{1}{2}, \quad \lambda \to 0,$$

since $a_{12} > a_{11}$, $a_{22} < a_{21}$, and $a_{32} > a_{31}$. Similarly,

$$\mathbf{P}(X_2 < X_3) \to 1 - p_3, \quad \mathbf{P}(X_3 < X_1) \to 1 - p_1, \quad \sigma \to 0,$$

and

$$P_{X_1 X_2 X_3} \to \min\{1 - p_1, 1 - p_2, 1 - p_3\} = 1 - \max\{p_1, p_2, p_3\} > \frac{1}{2}, \quad \sigma \to 0.$$

Proof of Theorem 2. This result is established by analogy with Proposition 2. We use the representation

$$(X_1, X_2, X_3) = (a_{\nu 1}, a_{\nu 2}, a_{\nu 3}) + \lambda(\eta_1, \eta_2, \eta_3),$$

where the random variables η_j, $j = 1,2,3$, are continuous, independent, and identically distributed. Let G denote their distribution.

Under the condition $\nu = 1$, we have

$$U_j = \sum_{i=1}^{3} p_i G \left(\frac{a_{1j} - a_{ij}}{\lambda} + \eta_j \right), \quad j = 1,2,3,$$

and consequently,

$$(U_1, U_2, U_3) \xrightarrow{d} \left(p_1 U_0^1, p_1 + p_2 U_2^0, p_1 + p_2 + p_3 U_3^0 \right), \quad \lambda \to 0.$$

In other words, we obtain the uniform distribution on the parallelepiped $[0, p_1] \times [p_1, p_1 + p_2] \times [p_1 + p_2, 1]$ in the limit. The cases $\nu = 2,3$ are considered by analogy.

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